Different types of games

- Simultaneous games
- Sequential games, Stackelberg games: consists of a leader and a follower, the follower observes the leader’s quantity choice and choose action based on that.

\[
\min_{x_1 \in X_1} \left\{ f_1(x_1, x_2) \mid x_2 \in \arg \min_{y \in X_2} f_2(x_1, y) \right\}
\]  (1)
Motivations

• Why we are interested in games?
  Use cases in ML: GANs, adversarial training, and primal-dual RL.

• What is the problem?
  Simple gradient based methods are not working and we are looking for other optimization methods.
GANs from Binglin, Shashan, and Bhargav.
GANs

\[
\min_G \max_D V(G, D) \tag{2}
\]

\[
V(G, D) = \int_x p_{data}(x) \log(D(x)) dx + \int_z p_z(z) \log(1 - D(g(z))) dz \tag{3}
\]

- Equilibrium no longer consist of a single loss, hence nested optimization.
GAN optimization algorithm

- GAN optimization is based on gradient descent ascent (GDA).
- Update the discriminator by ascending gradient:

  \[ \nabla_{\theta_d} \frac{1}{m} \sum_{i=1}^{m} \left[ \log D(x^{(i)}) + \log (1 - D(G(z^{(i)}))) \right] \quad (4) \]

- Update the generator by descending gradient:

  \[ \nabla_{\theta_g} \frac{1}{m} \sum_{i=1}^{m} \log (1 - D(G(z^{(i)}))) \quad (5) \]
Convergence of Learning Dynamics in Stackelberg Games

T. Fiez, B. Chasnov, and L. J. Ratliff
Games setting

- They considered a sequential Stackelberg game (pure strategy: Stackelberg equilibrium).
- This game consists of a leader and a follower.
Finite-Time High-Probability Guarantees

The follower converges to:

$$P(\|x_{2,n} - z_n\| \leq \varepsilon, \forall n \geq \bar{n}|x_{2,n_0}, z_{n_0} \in B_{q_0}) \rightarrow 1 \quad (6)$$

where, $z_k = r(x_{1,k})$ and $r(x)$ is the implicit function.
Finite-Time High-Probability Guarantees

The leader converges to:

\[ P \left( \| x_{1,n} - x_1(\hat{t}_n) \| \leq \varepsilon, \forall n \geq \bar{n} | x_{n_0}, x_{n_0} \in B_{q_0} \right) \rightarrow 1 \] (7)

Take away point, we converge to a neighborhood of a Stackelberg equilibrium in finite-time, with a good probability!!
Conclusions

• Shows that there exist stable attractors of simultaneous gradient play that are Stackelberg equilibria and not Nash equilibria.
Conclusions

- Shows that there exist stable attractors of simultaneous gradient play that are Stackelberg equilibria and not Nash equilibria.
- A finite-time high probability bound for local convergence to a neighborhood of a stable Stackelberg equilibrium in general-sum games.
On Solving Minimax Optimization Locally: A Follow-the-Ridge Approach

Under blind review at ICLR 2020
Games Setting

- Differentiable sequential games,
- Two players,
- zero-sum, minimax,

\[
\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} f(x, y) \tag{8}
\]
How to solve minimax optimization?

- **Gradient descent-ascent (GDA)**
  - Problem 1. The goal is to converge to local minimax points, but GDA fails.
  - Problem 2. Strong rotation around fixed points. Requires small learning rate.

- **Follow-the-Ridge (FR), proposed by this paper.**
  - Solves both issues.
Follow the ridge (FR)

- GDA tends to drift away from the ridge.
- How to solve it?
  By definition, a local minimax has to lie on a ridge.
  So, follow the ridge!
FR algorithm

**Algorithm** Follow-the-Ridge (FR). Differences from gradient descent-ascent are shown in blue.

**Require:** Learning rate $\eta_x$ and $\eta_y$; number of iterations $T$.

1: for $t = 1, \ldots, T$ do
2: \[ x_{t+1} \leftarrow x_t - \eta_x \nabla_x f(x_t, y_t) \] \quad \triangleright \text{gradient descent}
3: \[ y_{t+1} \leftarrow y_t + \eta_y \nabla_y f(x_t, y_t) + \eta_x H_{yy}^{-1} H_{yx} \nabla_x f(x_t, y_t) \] \quad \triangleright \text{modified gradient ascent}
FR algorithm

- Minimax
- Ridge
- Gradient step of the leader $x$
- Gradient step of the follower $y$
- Correction term of FR
FR results

from the paper.
Conclusion

- It addresses the rotational behaviour of gradient dynamics and allows larger learning rate than GDA.
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- Standard acceleration techniques can be added.
Conclusion

• It addresses the rotational behaviour of gradient dynamics and allows larger learning rate than GDA.
• Standard acceleration techniques can be added.
• In general we were so hyped about using GD in neural networks because we knew they are converging, this method can be viewed as a similar way to think about GANs.